

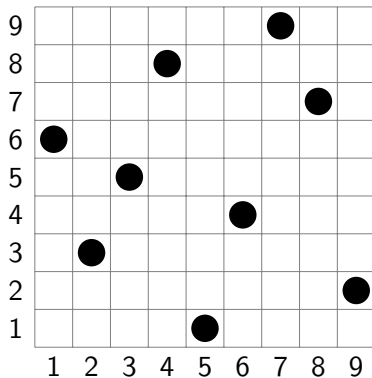
Some $1 \times n$ generalized grid classes are context-free

Robert Brignall Jakub Sliachan

Permutation Patterns 2018

View permutations as drawings

635814972



Enumerating permutation classes

Class

Collection of permutations closed under containment (if $\pi \in \mathcal{C}$, then all subpermutations $\sigma \subset \pi$ are also in \mathcal{C}).

Enumeration

Determining the number of permutations of each length in \mathcal{C} .

Context-free class

Definition

A class \mathcal{C} is *context-free* if it coincides with the first component of the system of equations

$$\begin{cases} \mathcal{S}_1 &= f_1(\mathcal{Z}, \mathcal{S}_1, \dots, \mathcal{S}_r) \\ &\vdots \\ \mathcal{S}_r &= f_r(\mathcal{Z}, \mathcal{S}_1, \dots, \mathcal{S}_r) \end{cases}$$

where f_i are constructors only involving $+$, \times , and $\mathcal{E} = \emptyset$.

Context-free class: example

$$\begin{aligned} \mathcal{S} &= \mathcal{Z} + \begin{array}{|c|} \hline \mathcal{S}_{\oplus} \\ \hline \end{array} \begin{array}{|c|} \hline \mathcal{S} \\ \hline \end{array} + \begin{array}{|c|} \hline \mathcal{S}_{\ominus} \\ \hline \end{array} \begin{array}{|c|} \hline \mathcal{S} \\ \hline \end{array} \\ \mathcal{S}_{\ominus} &= \mathcal{Z} + \begin{array}{|c|} \hline \mathcal{S} \\ \hline \end{array} \\ \mathcal{S}_{\oplus} &= \mathcal{Z} + \begin{array}{|c|} \hline \mathcal{S}_{\ominus} \\ \hline \end{array} \begin{array}{|c|} \hline \mathcal{S} \\ \hline \end{array} \end{aligned}$$

$$\begin{aligned} \mathcal{S} &= \mathcal{Z} + \mathcal{S}_{\oplus}\mathcal{S} + \mathcal{S}_{\ominus}\mathcal{S} \\ \mathcal{S}_{\ominus} &= \mathcal{Z} + \mathcal{S}_{\oplus}\mathcal{S} \\ \mathcal{S}_{\oplus} &= \mathcal{Z} + \mathcal{S}_{\ominus}\mathcal{S}, \end{aligned}$$

Context-free classes are nice

Many things are context-free, e.g.

finitely many simples \implies context-free

Shades of niceness

rational \subset algebraic $\subset D$ -finite $\subset D$ -algebraic \subset power series

Theorem (Chomsky-Schützenberger)

A combinatorial class \mathcal{C} that is context-free admits an algebraic generating function.

Grid classes

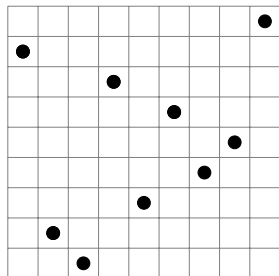
Definition

Permutation *grid class* is a permutation class. It consists of permutations that can be chopped up by vertical and horizontal lines into sub-permutations belonging to designated classes.

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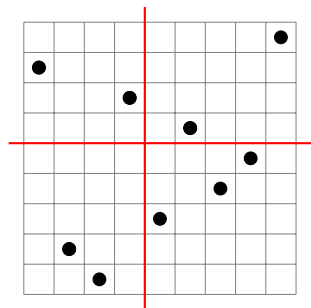


belongs to $\left[\begin{array}{cc} Av(12) & Av(21) \\ Av(12) & Av(21) \end{array} \right]$.

Grid classes

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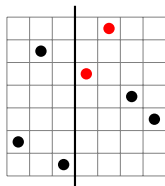
belongs to $\begin{bmatrix} \text{Av}(12) & \text{Av}(21) \\ \text{Av}(12) & \text{Av}(21) \end{bmatrix}$.

Example: where the trouble lies

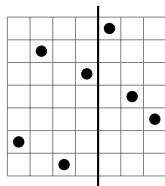
2615743 is in

$A_V(321)$	$A_V(12)$
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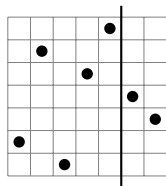
as witnessed by the middle two partitions.



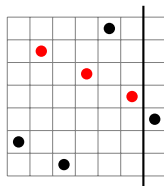
No!



Yes!



Yes!



No!

...also ...

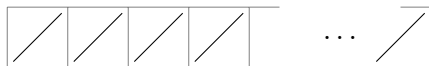
these have rational generating functions [AAB⁺13]

Geom

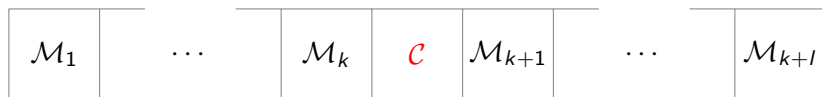
$$\left(\begin{array}{cccc|c} \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & \dots \mathcal{M} \\ \hline & & & & \\ \hline & & & & \\ & & \vdots & & \ddots \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & & \mathcal{M} \end{array} \right)$$

... and ...

generating functions conjectured for monotone increasing strips [Bev15b]



Today



Theorem

Let \mathcal{C} be a context-free permutation class that admits a combinatorial specification which tracks both the right-most and the left-most points. Let \mathcal{M}_i be a sequence of $n - 1$ monotone permutation classes. Then $\mathcal{M}_1 | \dots | \mathcal{M}_k | \mathcal{C} | \mathcal{M}_{k+1} | \dots | \mathcal{M}_{k+l}$ is a context-free permutation class that admits an algebraic generating function.

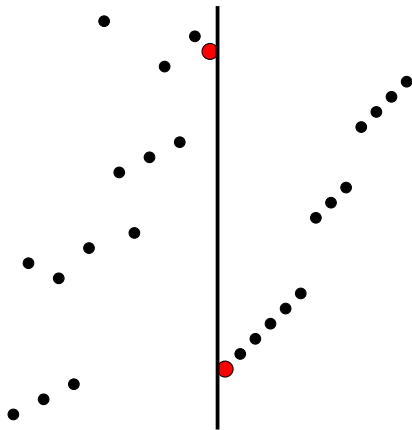
Leftmost gridlines

Griddable \rightarrow gridded

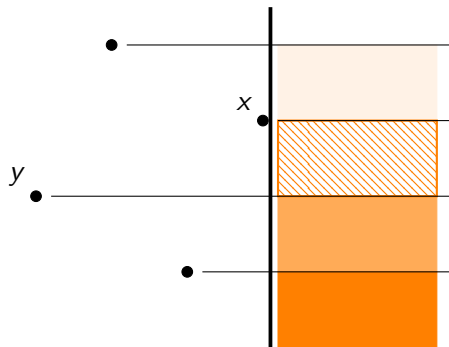
Convention:

Let π be a permutation from $\mathcal{C}_1|\mathcal{C}_2$. The gridline in π is chosen to be the left-most possible. I.e. if it was any further left, the sub-permutation to the right of it would not belong to the designated class \mathcal{C}_2 .

Leftmost gridlines: example $\mathcal{C}|A_V(21)$



Gaps associated with points



The *gap associated with x* is the space on the RHS below x and above the next point below it on the LHS.

What we want to do: example

Enumerate $A_V(21|21|21)$. Append cells from left to right.

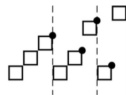
1. Start with a single increasing sequence on the LHS.



2. Now append stuff on the RHS.



3. Finally, append the third cell.



Tracking the rightmost point

The rightmost point of \mathcal{C} is critical. So pick the combinatorial specification of \mathcal{C} that tracks the rightmost point.

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$$\mathcal{S}^* = \mathcal{Z}^* + \begin{array}{|c|} \hline \mathcal{S}^* \\ \hline \end{array} + \begin{array}{|c|} \hline \mathcal{S}_\ominus \\ \hline \end{array} + \begin{array}{|c|} \hline \mathcal{S}_\oplus \\ \hline \end{array} + \begin{array}{|c|} \hline \mathcal{S}^* \\ \hline \end{array}$$

$$\mathcal{S} = \mathcal{Z} + \begin{array}{|c|} \hline \mathcal{S} \\ \hline \end{array} + \begin{array}{|c|} \hline \mathcal{S}_\ominus \\ \hline \end{array} + \begin{array}{|c|} \hline \mathcal{S} \\ \hline \end{array}$$

$$\mathcal{S}_\ominus = \mathcal{Z} + \begin{array}{|c|} \hline \mathcal{S} \\ \hline \end{array} + \begin{array}{|c|} \hline \mathcal{S}_\oplus \\ \hline \end{array}$$

$$\mathcal{S}_\oplus = \mathcal{Z} + \begin{array}{|c|} \hline \mathcal{S}_\ominus \\ \hline \end{array} + \begin{array}{|c|} \hline \mathcal{S} \\ \hline \end{array}$$

$$\mathcal{S}^* = \mathcal{Z}^* + \mathcal{S}_\oplus \mathcal{S}^* + \mathcal{S}^* \mathcal{S}_\ominus$$

$$\mathcal{S} = \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} + \mathcal{S} \mathcal{S}_\ominus$$

$$\mathcal{S}_\ominus = \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S}$$

$$\mathcal{S}_\oplus = \mathcal{Z} + \mathcal{S} \mathcal{S}_\ominus.$$

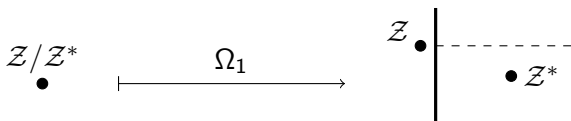
Operators

Consider Ω_1 , an operator that appends a single point on the right of a class $\mathcal{T}_m = X_1 \dots X_m$ (bottom to top).

$$\Omega_1(\mathcal{Z}) = \mathcal{Z}^* \mathcal{Z}$$

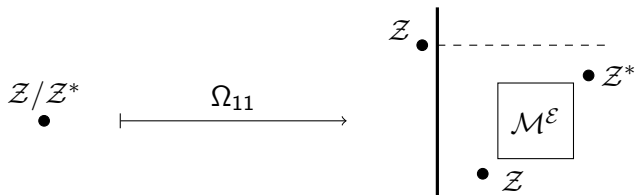
$$\Omega_1(\mathcal{Z}^*) = \mathcal{Z}^* \mathcal{Z}$$

$$\Omega_1(\mathcal{T}_m) = \begin{cases} \Omega_1(X_1^*)\Omega_0(X_2 \dots X_m) & \text{if } k = 1 \\ \Omega_1(X_1)\Omega_0(X_2 \dots X_m) + \Omega_0(X_1)\Omega_1(X_2 \dots X_m), & \text{if } k > 1. \end{cases}$$



The beast operator

Ω_{11} is the most involved operator – placing a sequence on the RHS with designated bottom and top point.

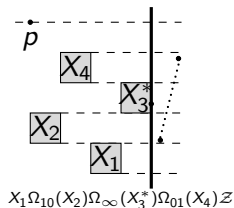
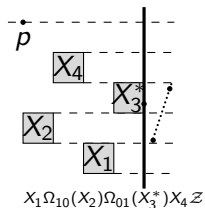
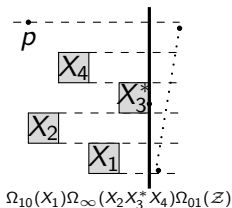
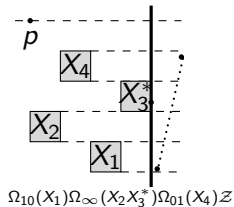
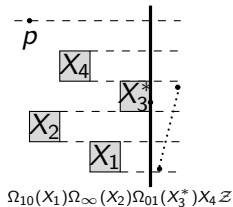
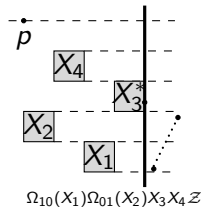
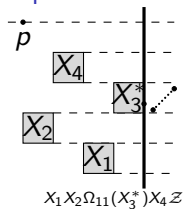
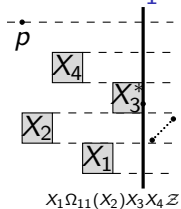
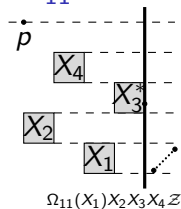


All operators

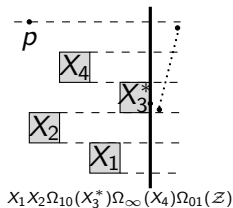
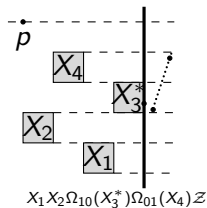
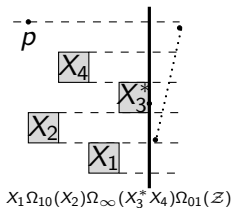
We need the following information captured when appending sequences on the RHS.

- ▶ Ω_0 : Nothing appended on the RHS.
- ▶ Ω_1 : Single point appended on the RHS (leftmost & rightmost coincide)
- ▶ Ω_∞ : Possibly empty sequence by itself.
- ▶ Ω_{10} : Point followed by a (possibly empty) sequence above.
- ▶ Ω_{01} : Point preceded by a (possibly empty) sequence below.
- ▶ Ω_{11} : Point followed by a (possibly empty) sequence followed by another point.

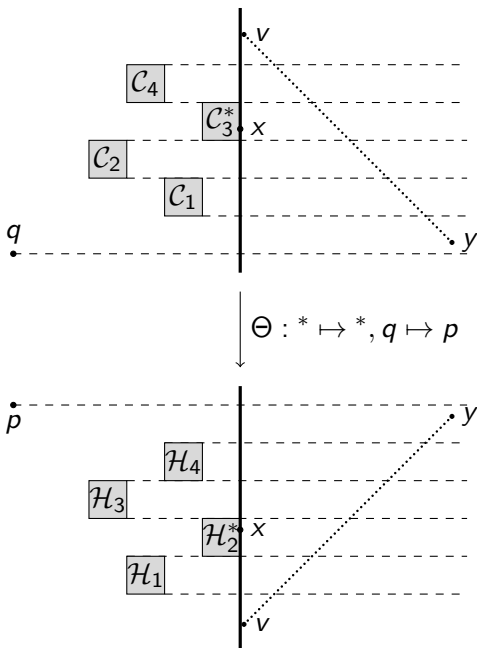
Apply Ω_{11} to a class $\mathcal{C} = X_1 X_2 X_3^* X_4$



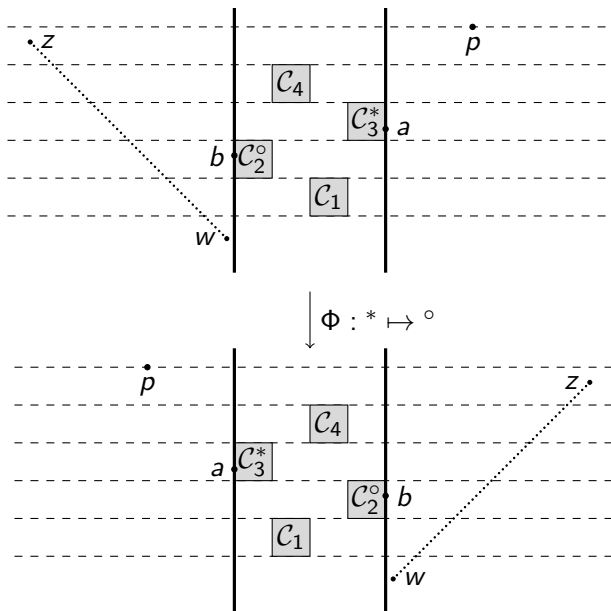
Apply Ω_{11} to a class $\mathcal{C} = X_1 X_2 X_3^* X_4$



Appending a monotone decreasing class



Depending on the left



Putting it all together

Consider $\mathcal{C}|_{\text{Av}(21)}$.

$$\mathcal{F} = \mathcal{E} + \mathcal{M} + \frac{\Omega_1(\bar{\mathcal{C}}^*) + \Omega_{11}(\bar{\mathcal{C}}^*)}{\mathcal{Z}}$$

- ▶ Either empty, or non-empty increasing, or non-empty \mathcal{C} next to non-empty $\text{Av}(21)$.
- ▶ Need phantom points, hence $\bar{\mathcal{C}}$.
- ▶ Need to track rightmost points only, so \mathcal{C}^* .
- ▶ Need to remove the phantom point after we're done, hence $1/\mathcal{Z}$ in the last term.

In general more complicated, but same ideas.

Things to notice

- ▶ algorithmic approach \rightarrow can be automated
- ▶ it's constructive: can enumerate (provide g.f. for) every such $1 \times n$ grid class
- ▶ rational? D-finite?
- ▶ $n \times m$ acyclic grid classes?
- ▶ etc.



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Geometric grid classes of permutations.

Transactions of the American Mathematical Society, 365(11):5859–5881, 2013.



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Growth rates of permutation grid classes, tours on graphs, and the spectral radius.

Transactions of the American Mathematical Society, 367(8):5863–5889, 2015.



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PhD thesis, The Open University, 2015.