Some $1 \times n$ generalized grid classes are context-free

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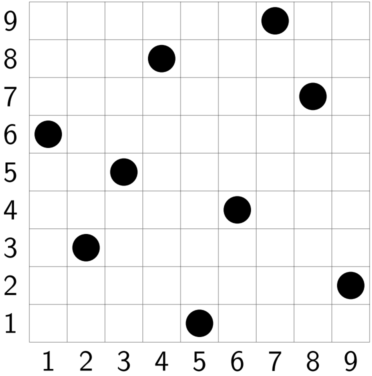
Permutation Patterns 2018

View permutations as drawings

635814972 $\longleftrightarrow$ 

Enumerating permutation classes

Class

Collection of permutations closed under containment (if $\pi \in \mathcal{C}$, then all subpermutations $\sigma \subset \pi$ are also in $\mathcal{C}$).

Enumeration

Determining the number of permutations of each length in $\mathcal{C}$.

Context-free class

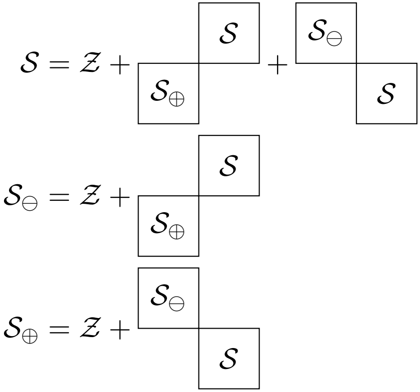
Definition

A class $\mathcal{C}$ is *context-free* if it coincides with the first component of the system of equations

$$\begin{cases} \mathcal{S}_1 & = f_1(\mathcal{Z}, \mathcal{S}_1, \ldots, \mathcal{S}_r) \\ & \vdots \\ \mathcal{S}_r & = f_r(\mathcal{Z}, \mathcal{S}_1, \ldots, \mathcal{S}_r) \end{cases}$$

where f_i are constructors only involving $+$, $\times$, and $\mathcal{E} = \emptyset$.

Context-free class: example



$$
\begin{aligned}
\mathcal{S} &= \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} + \mathcal{S}_\ominus \mathcal{S} \\
\mathcal{S}_\ominus &= \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} \\
\mathcal{S}_\oplus &= \mathcal{Z} + \mathcal{S}_\ominus \mathcal{S},
\end{aligned}
$$

Context-free classes are nice

Many things are context-free, e.g.

$$\text{finitely many simples} \implies \text{context-free}$$

Shades of niceness

$$\text{rational} \subset \text{algebraic} \subset D\text{-finite} \subset D\text{-algebraic} \subset \text{power series}$$

Theorem (Chomsky-Schützenberger)

A combinatorial class $\mathcal{C}$ that is context-free admits an algebraic generating function.

Grid classes

Definition

Permutation *grid class* is a permutation class. It consists of permutations that can be chopped up by vertical and horizontal lines into sub-permutations belonging to designated classes.

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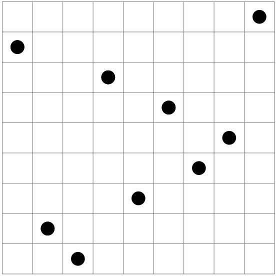

belongs to $\begin{bmatrix} \mathrm{Av}(12) & \mathrm{Av}(21) \\ \mathrm{Av}(12) & \mathrm{Av}(21) \end{bmatrix}$.

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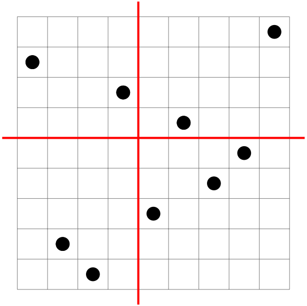


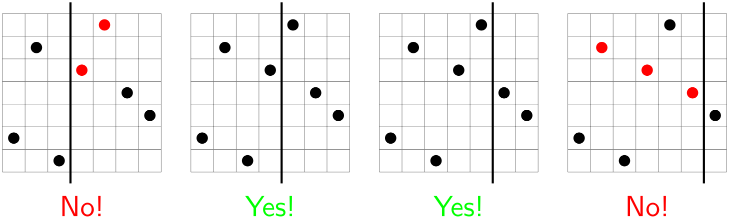
belongs to $\begin{bmatrix} \mathrm{Av}(12) & \mathrm{Av}(21) \\ \mathrm{Av}(12) & \mathrm{Av}(21) \end{bmatrix}$.

Example: where the trouble lies

2615743 is in

Av(321)	Av(12)

as witnessed by the middle two partitions.



We can't enumerate this

$\mathcal{C}_{11}$	$\mathcal{C}_{12}$	$\mathcal{C}_{13}$				$\mathcal{C}_{1m}$
$\mathcal{C}_{21}$	$\mathcal{C}_{22}$	$\mathcal{C}_{23}$				$\mathcal{C}_{2m}$
$\mathcal{C}_{31}$	$\mathcal{C}_{32}$	$\mathcal{C}_{33}$			$\cdots$	$\mathcal{C}_{3m}$
		$\vdots$			$\ddots$	
$\mathcal{C}_{n1}$	$\mathcal{C}_{n2}$	$\mathcal{C}_{n3}$				$\mathcal{C}_{nm}$

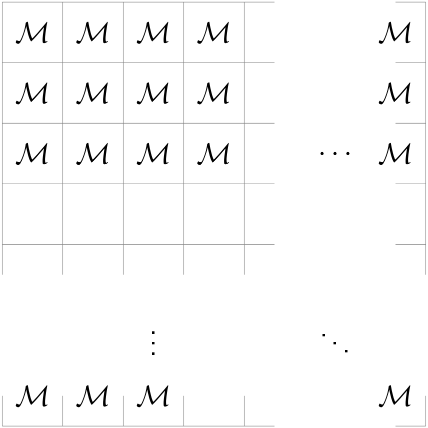
Even if $\mathcal{C}_{ij}$ are permutation classes that we CAN enumerate

…or this

$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$			$\mathcal{M}$
$\mathcal{M}$	$\mathcal{M}$	$\mathcal{C}$	$\mathcal{M}$			$\mathcal{M}$
$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$		$\cdots$	$\mathcal{M}$
		$\vdots$			$\ddots$	
$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$				$\mathcal{M}$

$\mathcal{M}$ monotone classes, $\mathcal{C}$ non-monotone class

...actually, not even this



$\mathcal{M}$ monotone classes

But! we know their growth rates = (spectral radius)^2 of the row-column graph [Bev15a].

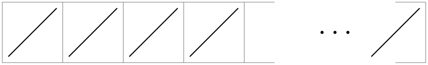
. . . also . . .

these have rational generating functions [AAB+13]

$$\mathrm{Geom}\begin{pmatrix} \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M} \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M} \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & \cdots & \mathcal{M} \\ & & & & & & \\ & & & & & & \\ & & \vdots & & & \ddots & \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & & & & \mathcal{M} \end{pmatrix}$$

. . . and . . .

generating functions conjectured for monotone increasing strips [Bev15b]



Today

$\mathcal{M}_1$		$\cdots$		$\mathcal{M}_k$	$\color{red}{\mathcal{C}}$	$\mathcal{M}_{k+1}$		$\cdots$		$\mathcal{M}_{k+l}$

Theorem

Let $\mathcal{C}$ be a context-free permutation class that admits a combinatorial specification which tracks both the right-most and the left-most points. Let $\mathcal{M}_i$ be a sequence of $n-1$ monotone permutation classes. Then $\mathcal{M}_1|\dots|\mathcal{M}_k|\mathcal{C}|\mathcal{M}_{k+1}|\dots|\mathcal{M}_{k+\ell}$ is a context-free permutation class that admits an algebraic generating function.

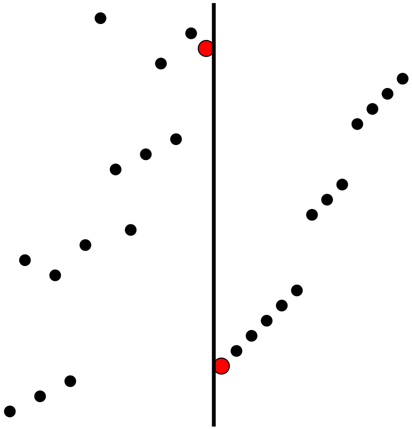
Leftmost gridlines

Griddable $\rightarrow$ gridded

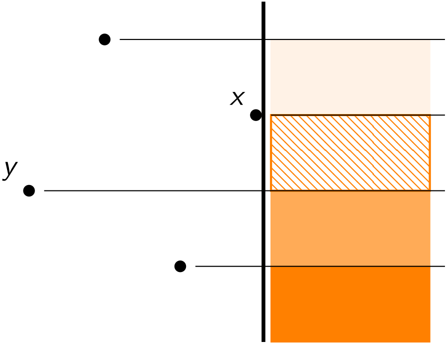
Convention:

Let π be a permutation from $\mathcal{C}_1|\mathcal{C}_2$. The gridline in π is chosen to be the left-most possible. I.e. if it was any further left, the sub-permutation to the right of it would not belong to the designated class $\mathcal{C}_2$.

Leftmost gridlines: example $\mathcal{C}|\mathrm{Av}(21)$



Gaps associated with points



The *gap associated with* x is the space on the RHS below x and above the next point below it on the LHS.

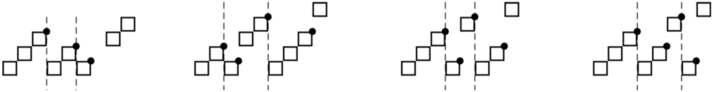
What we want to do: example

Enumerate $\mathrm{Av}(21|21|21)$. Append cells from left to right.

1. Start with a single increasing sequence on the LHS.

2. Now append stuff on the RHS.

3. Finally, append the third cell.



Tracking the rightmost point

The rightmost point of $\mathcal{C}$ is critical. So pick the combinatorial specification of $\mathcal{C}$ that tracks the rightmost point.

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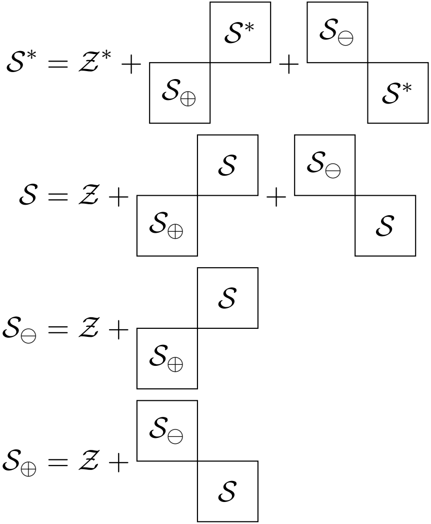


$$\begin{aligned}
\mathcal{S}^* &= \mathcal{Z}^* + \mathcal{S}_\oplus \mathcal{S}^* + \mathcal{S}^* \mathcal{S}_\ominus \\
\mathcal{S} &= \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} + \mathcal{S} \mathcal{S}_\ominus \\
\mathcal{S}_\ominus &= \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} \\
\mathcal{S}_\oplus &= \mathcal{Z} + \mathcal{S} \mathcal{S}_\ominus .
\end{aligned}$$

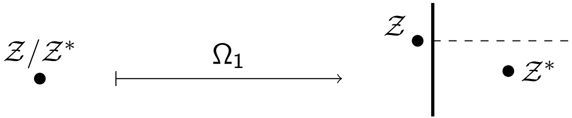
Operators

Consider Ω_1, an operator that appends a single point on the right of a class $\mathcal{T}_m = X_1 \ldots X_m$ (bottom to top).

$$\Omega_1(\mathcal{Z}) = \mathcal{Z}^*\mathcal{Z}$$

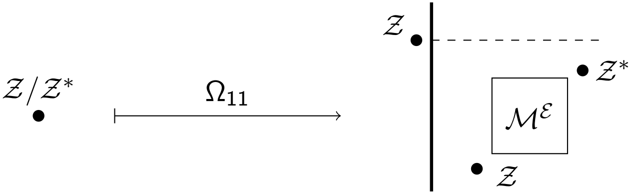
$$\Omega_1(\mathcal{Z}^*) = \mathcal{Z}^*\mathcal{Z}$$

$$\Omega_1(\mathcal{T}_m) = \begin{cases} \Omega_1(X_1^*)\Omega_0(X_2 \cdots X_m) & \text{if } k = 1 \\ \Omega_1(X_1)\Omega_0(X_2 \cdots X_m) + \Omega_0(X_1)\Omega_1(X_2 \cdots X_m), & \text{if } k > 1. \end{cases}$$



The beast operator

Ω_{11} is the most involved operator – placing a sequence on the RHS with designated bottom and top point.

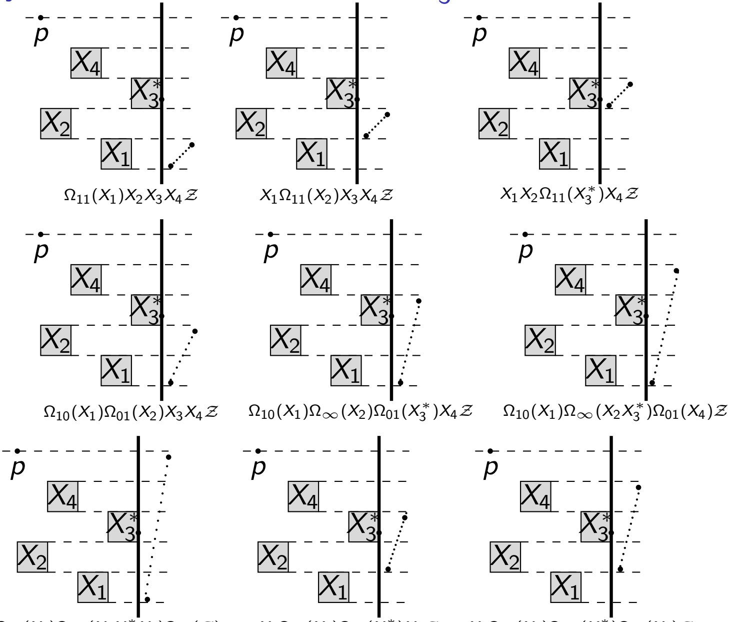


All operators

We need the following information captured when appending sequences on the RHS.

- Ω_0: Nothing appended on the RHS.
- Ω_1: Single point appended on the RHS (leftmost & rightmost coincide)
- Ω_∞: Possibly empty sequence by itself.
- Ω_{10}: Point followed by a (possibly empty) sequence above.
- Ω_{01}: Point preceded by a (possibly empty) sequence below.
- Ω_{11}: Point followed by a (possibly empty) sequence followed by another point.

Apply Ω_{11} to a class $\mathcal{C} = X_1 X_2 X_3^* X_4$

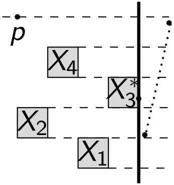


$\Omega_{11}(X_1)X_2X_3X_4\mathcal{Z}$ $\quad$ $X_1\Omega_{11}(X_2)X_3X_4\mathcal{Z}$ $\quad$ $X_1X_2\Omega_{11}(X_3^*)X_4\mathcal{Z}$

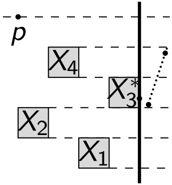
$\Omega_{10}(X_1)\Omega_{01}(X_2)X_3X_4\mathcal{Z}$ $\quad$ $\Omega_{10}(X_1)\Omega_{\infty}(X_2)\Omega_{01}(X_3^*)X_4\mathcal{Z}$ $\quad$ $\Omega_{10}(X_1)\Omega_{\infty}(X_2X_3^*)\Omega_{01}(X_4)\mathcal{Z}$

$\Omega_{10}(X_1)\Omega_{\infty}(X_2X_3^*X_4)\Omega_{01}(\mathcal{Z})$ $\quad$ $X_1\Omega_{10}(X_2)\Omega_{01}(X_3^*)X_4\mathcal{Z}$ $\quad$ $X_1\Omega_{10}(X_2)\Omega_{\infty}(X_3^*)\Omega_{01}(X_4)\mathcal{Z}$

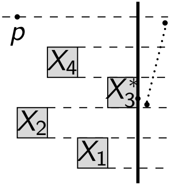
Apply Ω_{11} to a class $\mathcal{C} = X_1 X_2 X_3^* X_4$



$X_1 \Omega_{10}(X_2) \Omega_\infty (X_3^* X_4) \Omega_{01}(\mathcal{Z})$

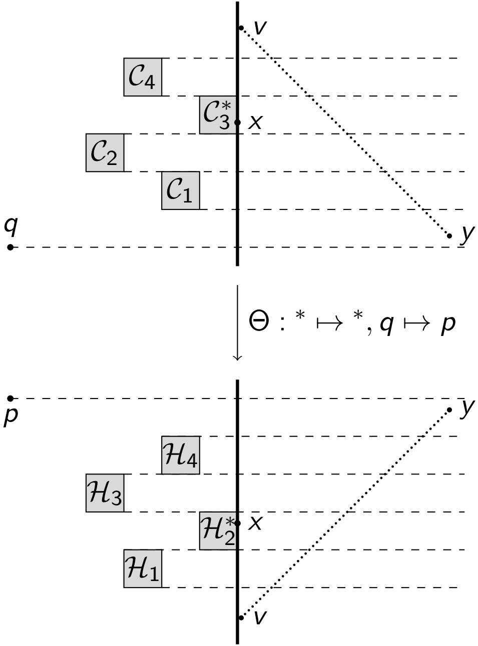


$X_1 X_2 \Omega_{10}(X_3^*) \Omega_{01}(X_4) \mathcal{Z}$

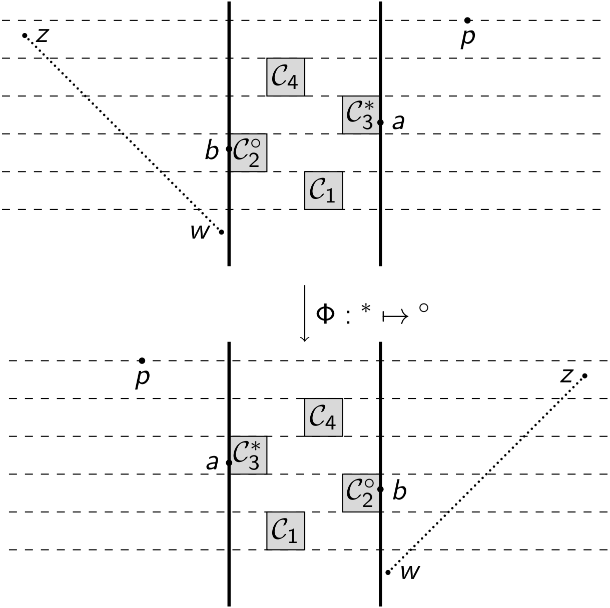


$X_1 X_2 \Omega_{10}(X_3^*) \Omega_\infty (X_4) \Omega_{01}(\mathcal{Z})$

Appending a monotone decreasing class



Appending on the left



Putting it all together

Consider $\mathcal{C}|\mathrm{Av}(21)$.

$$\mathcal{F} = \mathcal{E} + \mathcal{M} + \frac{\Omega_1(\overline{\mathcal{C}}^*) + \Omega_{11}(\overline{\mathcal{C}}^*)}{\mathcal{Z}}$$

- Either empty, or non-empty increasing, or non-empty $\mathcal{C}$ next to non-empty $\mathrm{Av}(21)$.
- Need phantom points, hence $\overline{\mathcal{C}}$.
- Need to track rightmost points only, so $\mathcal{C}^*$.
- Need to remove the phantom point after we're done, hence $1/\mathcal{Z}$ in the last term.

In general more complicated, but same ideas.

Things to notice

- algorithmic approach $\rightarrow$ can be automated
- it's constructive: can enumerate (provide g.f. for) every such $1 \times n$ grid class
- rational? D-finite?
- $n \times m$ acyclic grid classes?
- etc.

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On the growth of permutation classes.
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