Catalan classes next to monotone ones

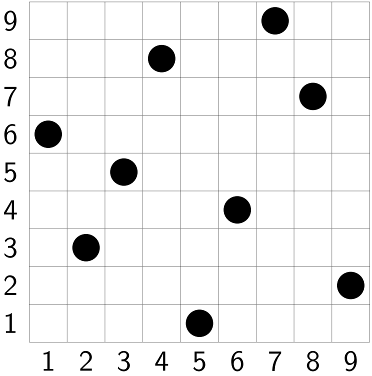
Jakub Sliačan (joint work with Robert Brignall)

British Combinatorial Conference 2017

arXiv:1611.05370

View permutations/patterns as drawings

635814972 $\longleftrightarrow$



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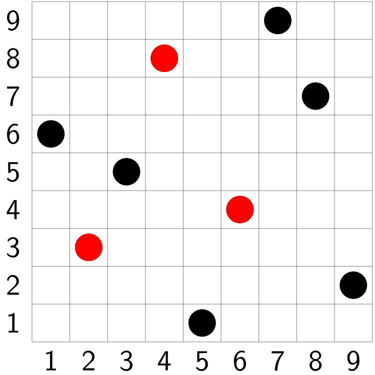
635814972 $\longleftrightarrow$ 

containement: 132 $\subset$ 635814972 (only relative order matters)

Enumerating permutation classes

Class

Collection of permutations closed under containment (if $\pi \in \mathcal{C}$, then all subpermutations $\sigma \subset \pi$ are also in $\mathcal{C}$)

Catalan class

A class of permutations that avoid one of the length 3 patterns: 123,132,213,231,312,321.

Monotone class $\mathcal{M}$

A class of permutations that avoid one of the length 2 patterns: 12,21.

Enumeration

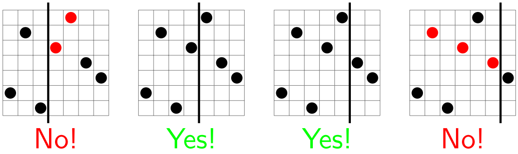
Determining the number of permutations of each length in $\mathcal{C}$

$\mathrm{Av}(abc|xy) =$ | Cat | $\mathcal{M}$ |

Let $\mathcal{C}_1, \mathcal{C}_2$ be permutation classes. Their *juxtaposition* $\mathcal{C} = \mathcal{C}_1|\mathcal{C}_2$ is the class of all permutations that can be partitioned such that the left part is a pattern from $\mathcal{C}_1$ and the right part is the pattern from $\mathcal{C}_2$.

Interested in: $\mathcal{C}_1 =$ Catalan class, $\mathcal{C}_2 =$ Monotone class.

Example: $2615743 \in \mathrm{Av}(321|12)$, witnessed by the middle two partitions.



Today

$$\begin{array}{ccc}
\mathrm{Av}(213|21),\ \underline{\mathbf{Av(231|12)}} & \overset{\theta}{\longleftrightarrow} & \underline{\mathrm{Av}(321|12)},\ \mathrm{Av}(123|21) \\
\mathrm{Av}(123|12),\ \underline{\mathbf{Av(321|21)}} & \overset{\psi}{\longleftrightarrow} & \underline{\mathrm{Av}(231|21)},\ \mathrm{Av}(213|12) \\
\mathrm{Av}(132|12),\ \underline{\mathbf{Av(312|21)}} & \overset{\phi}{\longleftrightarrow} & \underline{\mathrm{Av}(312|12)},\ \mathrm{Av}(132|21)
\end{array}$$

Enumerated by Bevan and Miner, respectively

Enumerated

Bijections θ, ψ, ϕ between underlined classes

Why these juxtapositions?

Because they show up, e.g.

- Bevan enumerated $\mathrm{Av}(231|12)$ (or its symmetry) as a step to enumerating $\mathrm{Av}(4213, 2143)$.
- Miner enumerated $\mathrm{Av}(123|21)$ (or its symmetry) as a step to enumerating $\mathrm{Av}(4123, 1243)$.

Because they are “simplest” grid classes

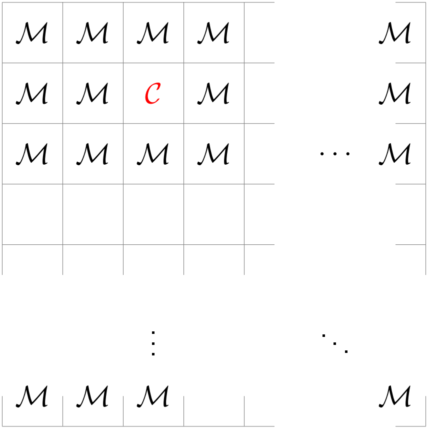
- Murphy, Vatter (2003)
- Albert, Atkinson, and Brignall (2011)
- Vatter, Watton (2011)
- Brignall (2012)
- Albert, Atkinson, Bouvel, Ruškuc, and Vatter (2013)
- Bevan (2016)

We can't enumerate this

$\mathcal{C}_{11}$	$\mathcal{C}_{12}$	$\mathcal{C}_{13}$				$\mathcal{C}_{1m}$
$\mathcal{C}_{21}$	$\mathcal{C}_{22}$	$\mathcal{C}_{23}$				$\mathcal{C}_{2m}$
$\mathcal{C}_{31}$	$\mathcal{C}_{32}$	$\mathcal{C}_{33}$			$\cdots$	$\mathcal{C}_{3m}$
		$\vdots$			$\ddots$	
$\mathcal{C}_{n1}$	$\mathcal{C}_{n2}$	$\mathcal{C}_{n3}$				$\mathcal{C}_{nm}$

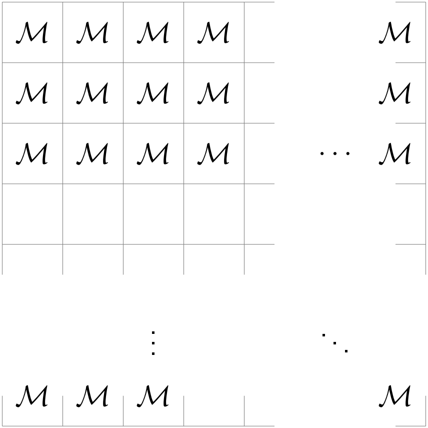
Even if $\mathcal{C}_{ij}$ are permutation classes that we CAN enumerate

. . . or this



$\mathcal{M}$ monotone classes, $\mathcal{C}$ non-monotone class

. . . actually, not even this



$\mathcal{M}$ monotone classes

But! we know their growth rates = (spectral radius)^2 of the row-column graph [Bev15a].

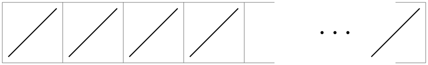
. . . also . . .

these have rational generating functions [AAB+13]

$$\mathrm{Geom}\begin{pmatrix} \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M} \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M} \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & \cdots & \mathcal{M} \\ & & & & & & \\ & & \vdots & & & \ddots & \\ \mathcal{M} & \mathcal{M} & \mathcal{M} & & & & \mathcal{M} \end{pmatrix}$$

. . . and . . .

generating functions conjectured for monotone increasing strips [Bev15b]



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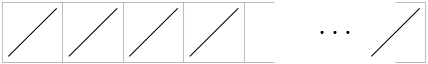

Idea: be less ambitious

So...

Enumerate juxtapositions of monotone and Catalan cells

We'll look at the blue parts

$$\mathrm{Av}(213|21),\ \underline{\mathbf{Av(231|12)}} \quad \overset{\theta}{\longleftrightarrow} \quad \mathrm{Av}(123|21),\ \underline{\mathrm{Av}(321|12)}$$
$$\mathrm{Av}(123|12),\ \underline{\mathbf{Av(321|21)}} \quad \overset{\psi}{\longleftrightarrow} \quad \mathrm{Av}(213|12),\ \underline{\mathrm{Av}(231|21)}$$
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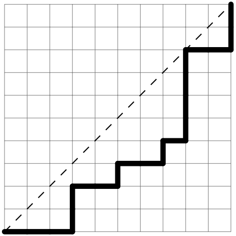
Dyck paths

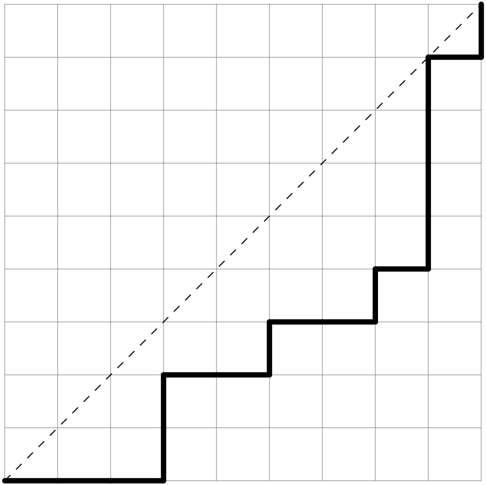
Dyck path

A Dyck path of length $2n$ is a path on the integer grid from top right to bottom left. Each step is either Down (D) or Left (L) and the path stays below the diagonal.

Example



231-avoiders and Dyck paths



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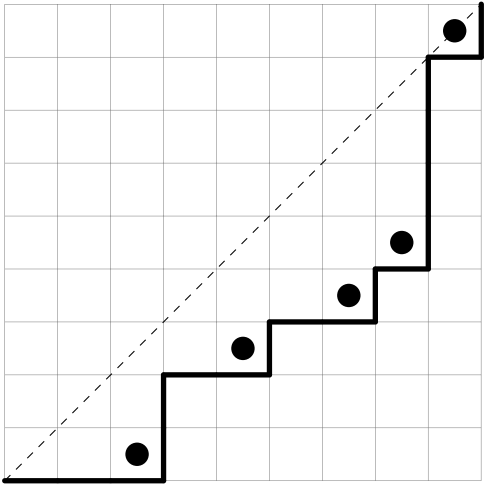

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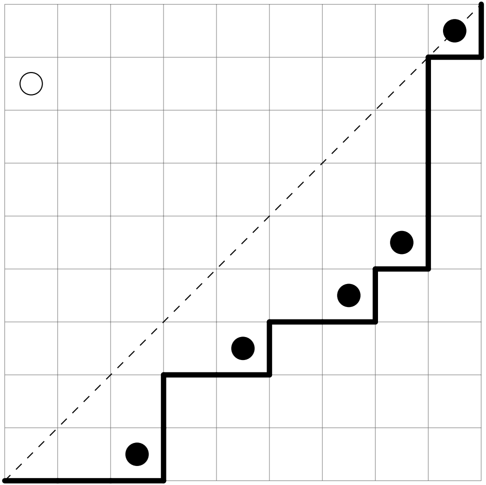

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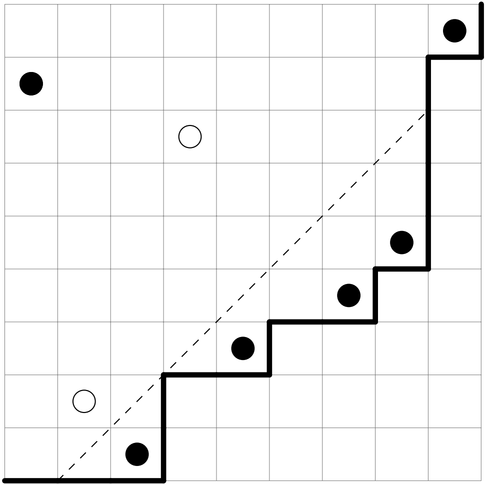

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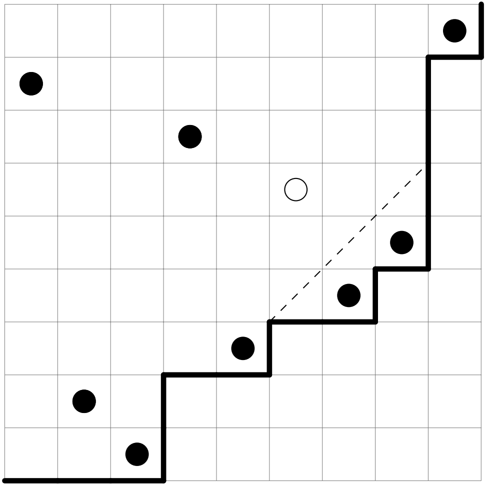

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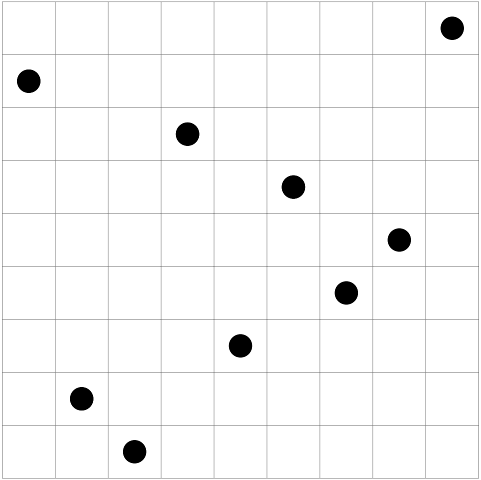

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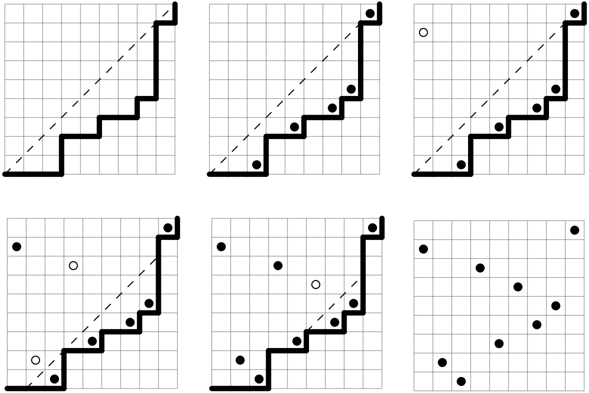
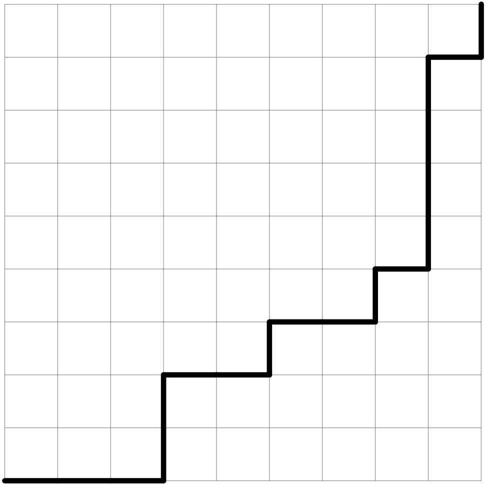


321-avoiders and Dyck paths



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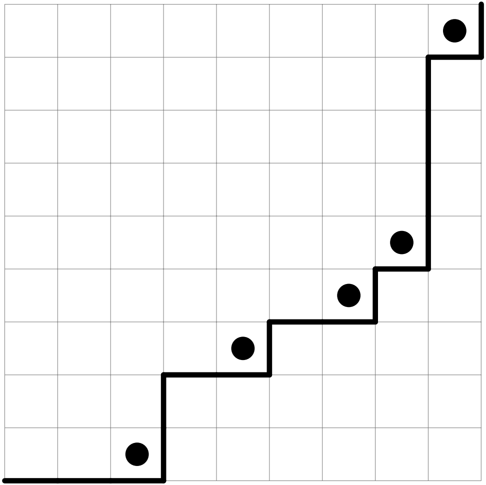

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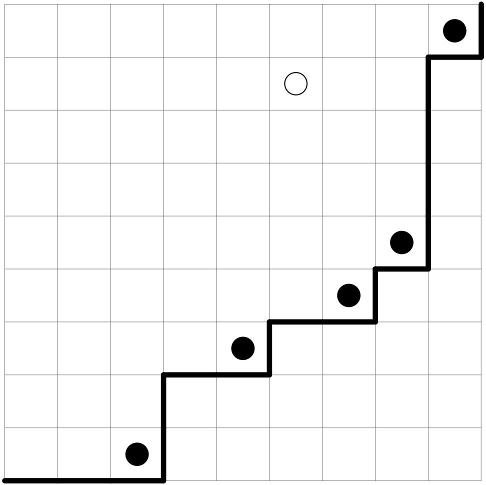

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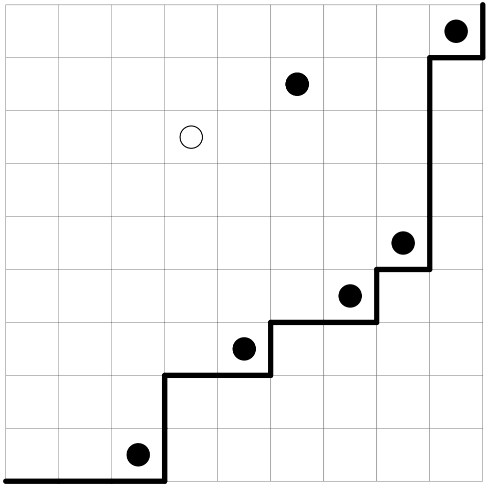

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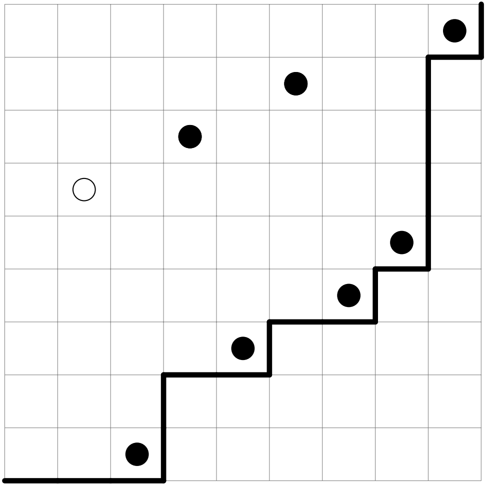

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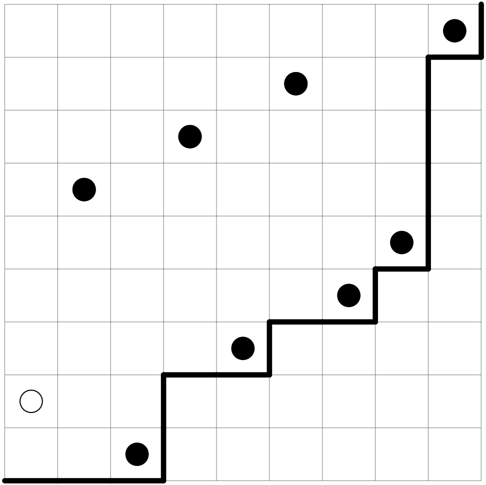

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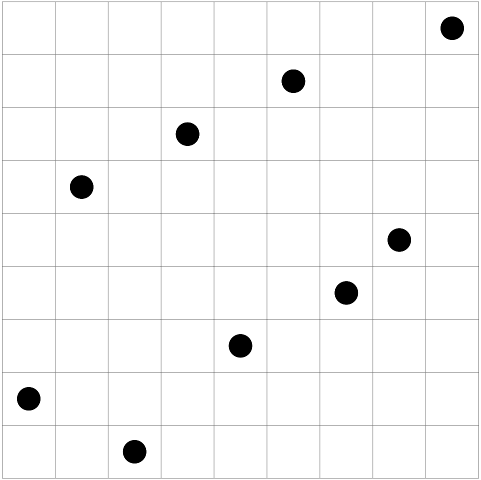

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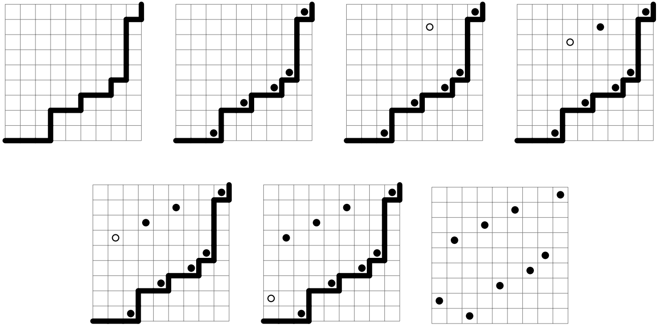


Context-free grammars

Definition

A context-free grammar (CFG) is a formal grammar that describes a language consisting of only those words which can be obtained from a starting string by repeated use of permitted production rules/substitutions.

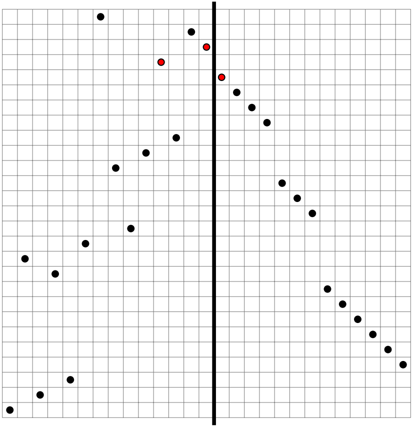
Example: Catalan class by itself (as a CFG)

- **variables:** C
- **characters:** $\epsilon, \mathsf{D}, \mathsf{L}$
- **relations:** $\mathsf{C} \to \epsilon \mid \mathsf{DCLC}$

This gives the following equation:

$$c = 1 + zc^2.$$

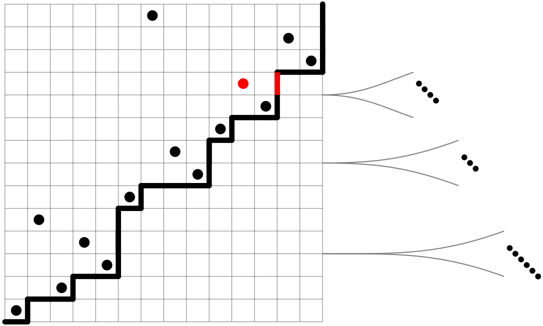
$Av(231|12)$ – gridline greedily right



griddable $\rightarrow$ gridded

$Av(231|12)$ – decorating Dyck paths

- insert point sequences under vertical steps
- first sequence (from top) under first vertical step after a horizontal step occured – first 12 occured



$\mathrm{Av}(231|12)$ – context-free grammar

- L – left step
- D – down step before any left steps occured
- **D** – down step after left step already occured

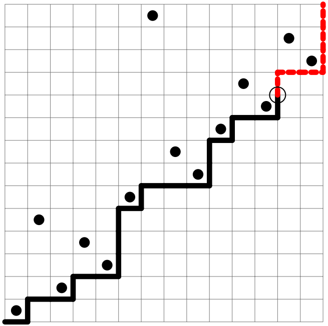
We denote by **C** a Dyck path over letters L and **D**, while C is a standard Dyck path over L and D.

$$\mathsf{S} \to \epsilon \mid \mathsf{DSL}\mathbf{C}$$
$$\mathbf{C} \to \epsilon \mid \mathbf{DC}\mathsf{L}\mathbf{C}$$

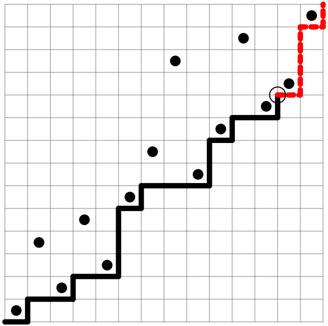
$$s = 1 + zs\mathbf{c}$$
$$\mathbf{c} = 1 + tz\mathbf{c}^2$$

$\mathrm{Av}(321|21)$ and $\mathrm{Av}(312|21)$ "similar".

Articulation point



(a) in $\mathrm{Av}(231)$



(b) in $\mathrm{Av}(321)$

common black part, unique red parts

Bijection $\theta : \mathrm{Av}(231|12) \to \mathrm{Av}(321|12)$

Idea

Choose a good bijection $\theta_0 : \mathrm{Av}(231) \to \mathrm{Av}(321)$. Then extend it to θ by preserving the RHS. □

Bijection $\phi : \mathrm{Av}(312|21) \to Av(312|12)$

Dyck paths $\mathcal{P}$ representing $\mathrm{Av}(312)$.

Recipe

1. Decompose $\mathcal{P}$ into excursions: $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_k$.
2. Identify *middle* part $\mathcal{P}_i$. Where pts on the RHS start.
3. Construct $\mathcal{P}'$ as: $\mathcal{P}_{i+1} \oplus \cdots \oplus \mathcal{P}_n \oplus \mathcal{P}_i \oplus \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_{i-1}$
4. Substitute $\mathcal{P}_i'$ for $\mathcal{P}_i$, where the order of vertical steps in $\mathcal{P}_i'$ is reversed (together with sequences of points on the RHS that go with those vertical steps).

Reversible and resulting Dyck path corresponds to a permutation from $\mathrm{Av}(312|12)$.

Summary

$$\begin{array}{ccc}
\mathrm{Av}(213|21),\ \underline{\mathbf{Av(231|12)}} & \stackrel{\theta}{\longleftrightarrow} & \mathrm{Av}(123|21),\ \underline{\mathrm{Av}(321|12)} \\
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\end{array}$$

Next

- non-Catalan juxtaposed with monotone
- iterated juxtapositions of monotone
- 2-dim monotone grid classes without cycles

M. H. Albert, M. D. Atkinson, M. Bouvel, N. Ruškuc, and V. Vatter.
Geometric grid classes of permutations.
Transactions of the American Mathematical Society, 365(11):5859–5881, 2013.

D. I. Bevan.
Growth rates of permutation grid classes, tours on graphs, and the spectral radius.
Transactions of the American Mathematical Society, 367(8):5863–5889, 2015.

D. I. Bevan.
On the growth of permutation classes.
PhD thesis, The Open University, 2015.